THE PACKING PROPERTY

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ABSTRACT. A clutter (V, E) *packs* if the smallest number of vertices needed to intersect all the edges (i.e. a minumum transversal) is equal to the maximum number of pairwise disjoint edges (i.e. a maximum matching). This terminology is due to Seymour 1977. A clutter is *minimally nonpacking* if it does not pack but all its minors pack. An $m \times n$ 0,1 matrix is *minimally nonpacking* if it is the edge-vertex incidence matrix of a minimally nonpacking clutter. Minimally nonpacking matrices can be viewed as the counterpart for the set covering problem of minimally imperfect matrices for the set packing problem. This paper proves several properties of minimally nonpacking clutters and matrices.

1. INTRODUCTION

A *clutter* $\mathcal{C}$ is a pair $\big(V(\mathcal{C}), E(\mathcal{C})\big)$, where $V(\mathcal{C})$ is a finite set and $E(\mathcal{C}) = \{S_1, \ldots, S_m\}$ is a family of subsets of $V(\mathcal{C})$ with the property that $S_i \subseteq S_j$ implies $S_i = S_j$. The elements of $V(\mathcal{C})$ are the *vertices* of $\mathcal{C}$ and those of $E(\mathcal{C})$ are the *edges*. A *transversal* of $\mathcal{C}$ is a subset of vertices that intersects all the edges. A transversal is *minimal* if none of its proper subset is a transversal. A transversal is *minimum* if no transversal has smaller cardinality. Let $\tau(\mathcal{C})$ denote the cardinality of a minimum transversal. A clutter $\mathcal{C}$ *packs* if there exist $\tau(\mathcal{C})$ pairwise disjoint edges.

For $j \in V(\mathcal{C})$, the *contraction* $\mathcal{C}/j$ and *deletion* $\mathcal{C} \setminus j$ are clutters defined as follows: both have $V(\mathcal{C}) - \{j\}$ as vertex set, $E(\mathcal{C}/j)$ is the set of minimal elements of $\{S - \{j\} : S \in E(\mathcal{C})\}$ and $E(\mathcal{C} \setminus j) = \{S : j \notin S \in E(\mathcal{C})\}$. Contractions and deletions of distinct vertices can be performed sequentially, and it is well known that the result does not depend on the order.

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A clutter $\mathcal{D}$ obtained from $\mathcal{C}$ by deleting $I_d \subseteq V(\mathcal{C})$ and contracting $I_c \subseteq V(\mathcal{C})$, where $I_c \cap I_d = \emptyset$ and $I_c \cup I_d \neq \emptyset$, is a *minor* of $\mathcal{C}$ and is denoted by $\mathcal{C} \setminus I_d / I_c$.

Note that the property that $\mathcal{C}$ packs is not closed under minor taking. For example, consider the graph with four vertices $V = \{1,2,3,4\}$ and four edges $E = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\}\}$. This clutter packs: indeed, $\{1,2\}$ is a minimum transversal and $\{\{1,4\},\{2,3\}\}$ is a matching of cardinality two. However, the clutter obtained by deleting vertex 4 is a graph with three vertices and the three edges $\{\{1,2\},\{1,3\},\{2,3\}\}$. This clutter does not pack: minimum transversals have cardinality two while maximum matchings have cardinality one. This observation leads us to consider the following property: We say that a clutter $\mathcal{C}$ has the *packing property* if it packs and all its minors pack. A clutter is *minimally non packing* (mnp) if it does not pack but all its minors do. In this paper, we study mnp clutters.

These concepts can be described equivalently in terms of 0,1 matrices. An $m \times n$ 0,1 matrix A *packs* if the minimum number of columns needed to cover all the rows equals the maximum number of nonoverlapping rows, i.e.

$$
\begin{aligned}
& \min\{e\,x : Ax \geq e, \;\; x \in \{0,1\}^n\} \\
= \;\; & \max\{y\,e : yA \leq e, \;\; y \in \{0,1\}^m\},
\end{aligned}
\tag{1.1}
$$

where e denotes a vector of appropriate dimension all of whose components are equal to 1. Obviously, dominating rows play no role in this definition (row $A_{i.}$ *dominates* row $A_{k.}$, $k \neq i$, if $A_{ij} \geq A_{kj}$ for all j), so we assume without loss of generality that A contains no such row. That is, we assume that A is the edge-vertex incidence matrix of a clutter. Since the statement "A packs" is invariant upon permutation of rows and permutation of columns, we denote by $A(\mathcal{C})$ any 0,1 matrix that is the edge-vertex incidence matrix of clutter $\mathcal{C}$. Observe that contracting $j \in V(\mathcal{C})$ corresponds to setting $x_j = 0$ in the set covering constraints $A(\mathcal{C})x \geq e$ (since, in $A(\mathcal{C}/j)$, column j is removed as well as the resulting dominating rows), and deleting j corresponds to setting $x_j = 1$ (since, in $A(\mathcal{C} \setminus j)$, column j is removed as well as all rows with a 1 in column j). The packing property for A requires that equation (1.1) holds for the matrix A itself and all its minors. This concept is dual to the concept of perfection (Berge [1]). Indeed, one can define a perfect 0,1 matrix as follows. A 0,1 matrix is *perfect* if all its column submatrices A satisfy the equation

$$
\begin{aligned}
& \max\{e\,x : Ax \leq e, \;\; x \in \{0,1\}^n\} \\
= \;\; & \min\{y\,e : yA \geq e, \;\; y \in \{0,1\}^m\}.
\end{aligned}
$$

This definition involves "column submatrices" instead of "minors" since setting a variable to 0 or 1 in the set packing constraints $Ax \leq e$ amounts to considering a column submatrix of A (in the case of setting a variable to 0, this is obvious, and in the case of setting a variable to 1, the constraints $Ax \leq e$ may force other variables to 0, so all the corresponding columns of A are removed). Pursuing the analogy, mnp matrices are to the set covering problem what minimally imperfect matrices are to the set packing problem.

The 0,1 matrix A is *ideal* if the polyhedron $\{x \geq 0 : Ax \geq e\}$ is integral (Lehman [9]). If A is ideal, then so are all its minors [16]. The following result is a consequence of Lehman's work [10].

Theorem 1.1. *If A has the packing property, then A is ideal.*

The converse is not true, however. A famous example is the matrix Q_6 with 4 rows and 6 columns comprising all 0,1 column vectors with two 0's and two 1's. It is ideal but it does not pack. This is in contrast to Lovász's theorem [11] stating that A is perfect if and only if the polytope $\{x \geq 0 : Ax \leq e\}$ is integral.

The 0,1 matrix A has the *Max-Flow Min-Cut property* (or simply MFMC property) if the linear system $Ax \geq e$, $x \geq 0$ is totally dual integral (Seymour [16]). Specifically, let

$$\begin{aligned} \tau(A, w) &= \min\{wx : Ax \geq e, \ \ x \in \{0,1\}^n\}, \\ \nu(A, w) &= \max\{y\ e : yA \leq w, \ \ y \in Z_+^m\}. \end{aligned}$$

A has the MFMC property if $\tau(A, w) = \nu(A, w)$ for all $w \in Z_+^n$. Setting $w_j = 0$ corresponds to deleting column j and setting $w_j = +\infty$ to contracting j. So, if A has the MFMC property, then A has the packing property. Conforti and Cornuéjols [3] conjecture that the converse is also true.

Conjecture 1.2. *A clutter has the packing property if and only if it has the MFMC property.*

This conjecture for the packing property is the analog of the following version of Lovász's theorem [11]: A $0,1$ matrix A is perfect if and only if the linear system $Ax \leq e, \ \ x \geq 0$ is totally dual integral.

In Section 2, we show that this conjecture holds for diadic clutters. A clutter is *diadic* if its edges intersect its minimal transversals in at most two vertices (Ding [6]). In fact, we show the stronger result:

Theorem 1.3. *A diadic clutter is ideal if and only if it has the MFMC property.*

A clutter is said to be *minimally non ideal* (mni) if it is not ideal but all its minors are ideal. Theorem 1.1 implies that all minors of an mnp clutter are ideal. Therefore mnp clutters fall into two distinct classes, namely:

Remark 1.4. A minimally non packing clutter is either ideal or mni.

Sections 3 and 4 deal with ideal mnp clutters. Seymour [16] showed that Q_6 is the only ideal mnp clutter which is binary (a clutter is *binary* if its edges have an odd intersection with its minimal transversals). Aside from Q_6, only one ideal mnp clutter was known prior to this work, due to Schrijver [14]. We construct an infinite family of such mnp clutters in Section 4. The clutter Q_6, Schrijver's example and those in our infinite class all satisfy $\tau(\mathcal{C}) = 2$. We prove in Section 3 that all ideal mnp clutters with $\tau(\mathcal{C}) = 2$ share strong structural properties with Q_6.

A clutter $\mathcal{C}$ has the Q_6 *property* if $A(\mathcal{C})$ has four rows such that every column of $A(\mathcal{C})$ restricted to this set of rows contains two 0's and two 1's and, furthermore, each of the six such possible 0,1 vectors occurs at least once.

Theorem 1.5. *Every ideal mnp clutter $\mathcal{C}$ with $\tau(\mathcal{C}) = 2$ has the Q_6 property.*

Our motivation for studying the Q_6 property was an attempt to characterize and, if possible, to enumerate all ideal mnp clutters. Section 4 shows that there is a rich family of ideal mnp clutters $\mathcal{C}$ with $\tau(\mathcal{C}) = 2$. These clutters are best described in terms of the Q_6 property, which they all share by Theorem 1.5. We make the following conjecture and we prove later in this section that it implies Conjecture 1.2.

Conjecture 1.6. *If $\mathcal{C}$ is an ideal mnp clutter, then $\tau(\mathcal{C}) = 2$.*

The *blocker* $b(\mathcal{C})$ of a clutter $\mathcal{C}$ is the clutter with $V(\mathcal{C})$ as vertex set and the minimal transversals of $\mathcal{C}$ as edge set. For $I_d, I_c \subseteq V(\mathcal{C})$ with $I_d \cap I_c = \emptyset$, it is well known and easy to derive that $b(\mathcal{C} \setminus I_d/I_c) = b(\mathcal{C})/I_d \setminus I_c$.

Section 5 studies minimally non ideal mnp clutters. The clutter $\mathcal{J}_t$, for $t \geq 2$ integer, is given by $V(\mathcal{J}_t) = \{0, \dots, t\}$ and $E(\mathcal{J}_t) = \big\{\{1, \dots, t\}, \{0, 1\}, \{0, 2\}, \dots, \{0, t\}\big\}$. Given a mni matrix A, let $\bar{x}$ be any vertex of $\{x \geq 0 : Ax \geq e\}$ with fractional components. A maximal row submatrix $\bar{A}$ of A for which $\bar{A}\bar{x} = e$ is called a *core* of A. The next result is due to Lehman [10] (see also Padberg [13], Seymour [17]).

Theorem 1.7. *Let A be an $m \times n$ mni matrix, $B = b(A)$, $r = \tau(B)$ and $s = \tau(A)$. Then*

(i) *A (resp. B) has a unique core $\bar{A}$ (resp. $\bar{B}$).*

(ii) *$\bar{A}, \bar{B}$ are square matrices.*

Moreover, either $A = A(\mathcal{J}_t)$, $t \geq 2$, or the rows and columns of $\bar{A}$ can be permuted so that

(iii) *$\bar{A}\bar{B}^T = J + (rs - n)I$, with $rs \geq n + 1$.*

Here J denotes a square matrix filled with ones and I the identity matrix. Only three cores with $rs = n + 2$ are known and none with $rs \geq n + 3$. Nevertheless Cornuéjols and Novick [5] have constructed more than one thousand mni matrices from a single core with $rs = n + 2$. An *odd hole* C_k^2 is a clutter with $k \geq 3$ odd, $V(C_k^2) = \{1, \dots k\}$ and $E(C_k^2) = \big\{\{1,2\},\{2,3\},\dots,\{k-1,k\},\{k,1\}\big\}$. Odd holes and their blockers are mni with $rs = n + 1$ and Luetolf and Margot [12] give dozens of additional examples of cores with $rs = n + 1$ and $n \leq 17$. We prove the following theorem.

Theorem 1.8. *Let $A \neq A(\mathcal{J}_t)$ be an $m \times n$ mni matrix. If A is minimally non packing, then $rs = n + 1$.*

We conjecture that the condition $rs = n + 1$ is also sufficient.

Conjecture 1.9. *Let $A \neq A(\mathcal{J}_t)$ be an $m \times n$ mni matrix. Then A is minimally non packing if and only if $rs = n + 1$.*

Using a computer program, we were able to verify this conjecture for all known mni matrices with $n \leq 14$.

A clutter is *minimally non MFMC* if it does not have the MFMC property but all its minors do. Conjecture 1.2 states that these are exactly the mnp clutters. Although we cannot prove this conjecture, the next proposition shows that a tight link exists between minimally non MFMC and mnp clutters. The clutter $\mathcal{D}$ obtained by *replicating* element $j \in V(\mathcal{C})$ of $\mathcal{C}$ is defined as follows: $V(\mathcal{D}) = V(\mathcal{C}) \cup \{j'\}$ where $j' \notin V(\mathcal{C})$, and

$$E(\mathcal{D}) = E(\mathcal{C}) \cup \{S - \{j\} \cup \{j'\} : j \in S \in E(\mathcal{C})\}.$$

Element j' is called a *replicate* of j. Let e_j denote the j^{th} unit vector.

Remark 1.10. $\mathcal{D}$ packs if and only if $\tau(\mathcal{C}, e + e_j) = \nu(\mathcal{C}, e + e_j)$.

Remark 1.11. $\mathcal{D}$ is ideal if and only if $\mathcal{C}$ is: As $\mathcal{C}$ is a deletion minor of $\mathcal{D}$, if $\mathcal{D}$ is ideal then $\mathcal{C}$ is ideal [16]. Conversely, if $\mathcal{D}$ is not ideal, there exists a fractional extreme point z of the polyhedron $P_D = \{x \geq 0 : A(\mathcal{D})x \geq e\}$. Note that $z_j = z_{j'}$, otherwise the larger of the two can be reduced or incremented while retaining feasibility, a contradiction with z being an extreme point. Let $\bar{z}$ be the vector obtained by removing component $z_{j'}$ from z. If $\mathcal{C}$ is ideal, then $\bar{z}$ is a convex combination of integer extreme points of $\{x \geq 0 : A(\mathcal{C})x \geq e\}$. This convex combination extend to a convex combination of points in P_D generating z, a contradiction.

Proposition 1.12. *Let $\mathcal{C}$ be a minimally non MFMC clutter. We can construct a minimally non packing clutter $\mathcal{D}$ by replicating elements of $V(\mathcal{C})$.*

Proof. Let $w \in Z_+^n$ be chosen such that $\tau(\mathcal{C}, w) > \nu(\mathcal{C}, w)$ and $\tau(\mathcal{C}, w') = \nu(\mathcal{C}, w')$ for all $w' \in Z_+^n$ with $w' \leq w$ and $w'_j < w_j$ for at least one j. Note that $w_j > 0$ for all j, since otherwise some deletion minor of $\mathcal{C}$ does not have the MFMC property. Construct $\mathcal{D}$ by replicating $w_j - 1$ times every element $j \in V(\mathcal{C})$. We show that $\mathcal{D}$ is minimally non packing. By Remark 1.10, $\mathcal{D}$ does not pack. Let $\mathcal{D}' = \mathcal{D} \setminus I_d / I_c$ be any minor of $\mathcal{D}$. We claim that $\mathcal{D}'$ packs. If j or one of its replicates j' is in I_c then we can assume that j and all its replicates are in I_c, since each subset $D \in E(\mathcal{D})$ with $j' \in D$ contains a set $B \in E(\mathcal{D}/j)$, i.e. D is a dominating subset in $\mathcal{D}/j$. Then $\mathcal{D}'$ is a replication of a minor $\mathcal{C}'$ of $\mathcal{C}/j$. Since $\mathcal{C}'$ has the MFMC property, $\mathcal{D}'$ packs by Remark 1.10. Thus we can assume $I_c = \emptyset$. By the choice of w and Remark 1.10, if $I_d \neq \emptyset$ then $\mathcal{D}'$ packs. This proves the claim and therefore the proposition. □

Proposition 1.12 can be used to show that, if every ideal mnp clutter $\mathcal{C}$ satisfies $\tau(\mathcal{C}) = 2$, then the packing property and the MFMC property are the same.

Proposition 1.13. *Conjecture 1.6 implies Conjecture 1.2.*

Proof. Suppose there is a minimally non MFMC clutter $\mathcal{C}$ that has the packing property. By Theorem 1.1, $\mathcal{C}$ is ideal. By Proposition 1.12, there is a mnp clutter $\mathcal{D}$ with a replicated element j. Furthermore, by remark 1.11, $\mathcal{D}$ is ideal. Using Conjecture 1.6, $2 = \tau(\mathcal{D}) \leq \tau(\mathcal{D}/j)$. Since $\mathcal{D}/j$ packs, there are sets $S_1, S_2 \in E(\mathcal{D})$ with $S_1 \cap S_2 = \{j\}$. Because j is replicated in $\mathcal{D}$, we have a set $S'_1 = S_1 \cup \{j'\} - \{j\}$. Note that $j' \notin S_2$. But then $S'_1 \cap S_2 = \emptyset$, hence $\mathcal{D}$ packs, a contradiction. □

In Section 6, we introduce a new class of clutters called weakly binary. They can be viewed as a generalization of binary and of balanced clutters. (A 0,1 matrix is *balanced* if it does not have $A(C_k^2)$ as a submatrix, $k \geq 3$ odd, where as above C_k^2 denotes an odd hole. See [4] for a survey of balanced matrices). We say that a clutter $\mathcal{C}$ has an odd hole C_k^2 if $A(C_k^2)$ is a submatrix of $A(\mathcal{C})$. An odd hole C_k^2 of $\mathcal{C}$ is said to have a *non intersecting set* if $\exists S \in E(\mathcal{C})$ such that $S \cap V(C_k^2) = \emptyset$. A clutter is *weakly binary* if, in $\mathcal{C}$ and all its minors, all odd holes have non intersecting sets. Balanced clutters are trivially weakly binary and we show in Section 6 that binary clutters are also weakly binary.

Theorem 1.14. *Let $\mathcal{C}$ be weakly binary and minimally non MFMC. Then $\mathcal{C}$ is ideal.*

Note that, when $\mathcal{C}$ is binary, this theorem is an easy consequence of Seymour's theorem saying that a binary clutter has the MFMC property if and only if it does not have Q_6 as a minor [16]. Indeed, Seymour's theorem implies that the only binary clutter that is minimally non MFMC is Q_6, which is ideal. Observe also that Theorem 1.14 together with Conjecture 1.6, Proposition 1.13, and Theorem 1.5, would imply that a weakly binary clutter has the MFMC property if and only if it does not contain a minor with the Q_6 property.

2. General properties of ideal minimally non packing clutters

Let $\mathcal{C}$ be ideal and let $\tilde{\mathcal{C}}$ be the clutter with same vertex set as $\mathcal{C}$ and edge set containing those edges of $\mathcal{C}$ that intersect exactly once each minimum transversal of $\mathcal{C}$. In other words: $E(\tilde{\mathcal{C}}) = \{S \in E(\mathcal{C}) : |T \cap S| = 1 \text{ for every } T \in E\big(b(\mathcal{C})\big) \text{ with } |T| = \tau(\mathcal{C})\}$. Consider

$$\tau(\mathcal{C}) = \min\{ex : A(\mathcal{C})x \geq e, x \geq 0\} \tag{2.2}$$

$$= \max\{ye : yA(\mathcal{C}) \leq e, y \geq 0\}. \tag{2.3}$$

Let T be any transversal with $|T| = \tau(\mathcal{C})$ and let x be its incidence vector. Since $\mathcal{C}$ is ideal, x is an optimal solution to (2.2). Thus if $A_{i.}x > 1$, then by complementary slackness $y_i = 0$ for all optimal solutions to (2.3). Conversely if $A_{i.}x = 1$ for all optimal solutions x to (2.2), then, by [15] p.95 (36), there is an optimal solution y to (2.3) with $y_i > 0$. It follows,

Remark 2.1. $A(\tilde{\mathcal{C}})$ contains exactly the rows $A(\mathcal{C})_{i.}$ for which there is an optimum solution y to (2.3) with $y_i > 0$.

We start with a collection of properties that an ideal mnp clutter satisfies.

Proposition 2.2. *Let $\mathcal{C}$ be an ideal minimally non packing clutter. Then*

(i) $\forall i \in V(\mathcal{C}), \tau(\mathcal{C} \setminus i) = \tau(\mathcal{C}) - 1$.

(ii) $yA(\mathcal{C}) = e$ *for all optimum solutions to* $\max\{ye : yA(\mathcal{C}) \leq e, y \geq 0\}$.

(iii) $\tau(\mathcal{C}) = \tau(\tilde{\mathcal{C}})$.

(iv) $\forall S \in E(\mathcal{C}), \exists T \in E\big(b(\mathcal{C})\big)$ *such that* $|T - S| \leq \tau(\mathcal{C}) - 2$.

(v) $\forall S \in E(\tilde{\mathcal{C}}), \exists T \in E\big(b(\mathcal{C})\big)$ *with* $|T| > \tau(\mathcal{C})$ *such that* $|T - S| \leq \tau(\mathcal{C}) - 2$.

(vi) If two columns c^i, c^j *of* $A(\tilde{\mathcal{C}})$ *satisfy* $c^i \leq c^j$, *then* $c^i = c^j$.

(vii) $\forall i \in V(\mathcal{C}), \tau(\mathcal{C}/i) = \tau(\mathcal{C})$.

Proof.

(i): By definition of deletion, $\tau(\mathcal{C} \setminus i) \geq \tau(\mathcal{C}) - 1$. Since $\mathcal{C}$ is mnp, there is a family $\mathcal{F} = \{S_1, \ldots, S_{\tau(\mathcal{C}\setminus i)}\}$ of pairwise disjoint edges of $E(\mathcal{C} \setminus i)$. Since $\mathcal{F} \subseteq E(\mathcal{C})$ and $\mathcal{C}$ does not pack, $|\mathcal{F}| = \tau(\mathcal{C} \setminus i) < \tau(\mathcal{C})$. The result follows.

(ii): Follows from (i) by complementary slackness.

(iii): The equality $\tau(\mathcal{C}) = \tau(\tilde{\mathcal{C}})$ follows from

$$\begin{aligned}
\tau(\mathcal{C}) &= \min\{ex : A(\mathcal{C})x \geq e, x \geq 0\} = \max\{ye : yA(\mathcal{C}) \leq e, y \geq 0\} \\
&= \max\{\tilde{y}e : \tilde{y}A(\tilde{\mathcal{C}}) \leq e, \tilde{y} \geq 0\} = \min\{ex : A(\tilde{\mathcal{C}})x \geq e, x \geq 0\} \\
&\leq \tau(\tilde{\mathcal{C}}) \leq \tau(\mathcal{C}).
\end{aligned}$$

This first equality follows by the fact that $\mathcal{C}$ is ideal, the second and fourth equality by duality and the third from the fact that, by Remark 2.1, $y_i = 0$ for all rows of $A(\mathcal{C})$ which are not rows of $A(\tilde{\mathcal{C}})$.

(iv): If $\forall T \in E\big(b(\mathcal{C})\big), |T - S| \geq \tau(\mathcal{C}) - 1$, then $\tau(\mathcal{C} \setminus S) \geq \tau(\mathcal{C}) - 1$. $\mathcal{C}$ is mnp, therefore there is a family $\mathcal{F} = \{S_1, \ldots, S_{\tau(\mathcal{C})-1}\} \subseteq E(\mathcal{C} \setminus S)$ of pairwise disjoint edges. Hence, $\{S\} \cup \mathcal{F}$ is a family of $\tau(\mathcal{C})$ pairwise disjoint edges of $\mathcal{C}$, i.e. $\mathcal{C}$ packs, a contradiction.

(v): Let $S \in E(\tilde{C})$ and $T \in E\big(b(\mathcal{C})\big)$ with $|T| = \tau(\mathcal{C})$. Then by definition of $\tilde{\mathcal{C}}$, $|T - S| = |T| - |S \cap T| = \tau(\mathcal{C}) - 1$ and the result follows by (iv).

(vi): Assume that $c^i \leq c^j$ and $c^i_k < c^j_k$. By Remark 2.1, there is an optimal solution y with $y_k > 0$ to $\max\{ye : yA(\mathcal{C}) \leq e, y \geq 0\}$. Moreover, $y_\ell = 0$ for all rows l of $A(\mathcal{C})$ which are not rows of $A(\tilde{\mathcal{C}})$. It follows that $yA(\mathcal{C})_{.i} < yA(\mathcal{C})_{.j}$, a contradiction with (ii).

(vii): By (vi), $\exists S \in E(\tilde{\mathcal{C}})$ with $i \in S$. Suppose $\tau(\mathcal{C}/i) > \tau(\mathcal{C})$. We will show that $S = \{i\}$, a contradiction to (iv). Consider any $j \in V(\mathcal{C}) - \{i\}$. By (i) $\exists S_j \in E\big(b(\mathcal{C})\big)$ with $|S_j| = \tau(\mathcal{C})$ and $j \in S_j$. Since $\tau(\mathcal{C}/i) > \tau(\mathcal{C})$, we know $i \in S_j$. But by definition of $\tilde{\mathcal{C}}$, we have $1 = |S \cap S_j| = |\{i\}|$, hence $j \notin S$. □

Proposition 2.2 is sufficient to prove Theorem 1.3 stating that a diadic clutter is ideal if and only if it has the MFMC property.

Proof of Theorem 1.3: Since clutters with the MFMC property are ideal, it is sufficient to show that all ideal diadic clutters have the MFMC property. By contradiction, let $\mathcal{C}$ be an ideal diadic clutter which is minimally non MFMC. By Proposition 1.12, there is a mnp clutter $\mathcal{D}$ obtained by replicating elements of $\mathcal{C}$. Note that the property of being diadic is closed under replication thus $\mathcal{D}$ is diadic. By Proposition 2.2 (v), $\forall S \in E(\tilde{\mathcal{D}}), \exists T \in E\big(b(\mathcal{D})\big)$ with $|T| > \tau(\mathcal{D})$ such that $|T| - |S \cap T| \leq \tau(\mathcal{D}) - 2$, a contradiction to $|S \cap T| \leq 2$. □

3. The Q_6 property

We say that a clutter has the Q_6 *property*, if $V(\mathcal{C})$ can be partitioned into nonempty sets $I_1, \ldots, I_6$, such that there are edges $S_1, \ldots, S_4$ in $\mathcal{C}$ of the form:

$$S_1 = I_1 \cup I_3 \cup I_5, \qquad S_2 = I_1 \cup I_4 \cup I_6,$$
$$S_3 = I_2 \cup I_4 \cup I_5, \qquad S_4 = I_2 \cup I_3 \cup I_6.$$

Note that Q_6 trivially has the Q_6 property. Now we prove Theorem 1.5 stating that, if $\mathcal{C}$ is an ideal mnp clutter with $\tau(\mathcal{C}) = 2$, then $\mathcal{C}$ has the Q_6 property.

Proof of Theorem 1.5: Let A denote $A(\mathcal{C})$ and $\tilde{A}$ denote $A(\tilde{\mathcal{C}})$. Since $\tau(\mathcal{C}) = 2$, $\exists k, l \in V(\mathcal{C})$ such that $\{k, l\} \in E\big(b(\mathcal{C})\big)$. Let $K = \{i : \tilde{A}_{.i} = \tilde{A}_{.k}\}$ and $L = \{i : \tilde{A}_{.i} = \tilde{A}_{.l}\}$. Observe that, by definition of $\tilde{A}$, we have $\tilde{A}_{.k} + \tilde{A}_{.l} = e$. We claim that

$$\tau(\mathcal{C} \setminus K/L) > 1. \tag{3.4}$$

Assume that the claim is false, i.e. there exists a transversal S of $\mathcal{C}$ with $|S - K| \leq 1$ and $S \cap L = \emptyset$. Trivially, S is a transversal of $\tilde{\mathcal{C}}$. By Proposition 2.2 (iii), we have $\tau(\tilde{\mathcal{C}}) = \tau(\mathcal{C}) = 2$. Since $|S - K| \geq \tau(\tilde{\mathcal{C}} \setminus K) = \tau(\tilde{\mathcal{C}} \setminus i) \geq 1$ for any $i \in K$, we have that $S - K = \{t\}$ for some $t \in V(\mathcal{C}) - (K \cup L)$. Moreover, $\tilde{A}_{.t} \geq \tilde{A}_{.l}$. By Proposition 2.2 (vi), this inequality cannot be strict, and thus $\tilde{A}_{.t} = \tilde{A}_{.l}$. This implies $t \in L$, a contradiction.

Since $\mathcal{C} \setminus K/L$ packs, there exist $S_1, S_2 \in E(\mathcal{C})$ such that:

$$(S_1 \cup S_2) \cap K = \emptyset \quad \text{and} \quad (S_1 \cap S_2) \cap \big(V(\mathcal{C}) - (K \cup L)\big) = \emptyset. \tag{3.5}$$

By symmetry, we must also have sets $S_3, S_4 \in E(\mathcal{C})$ such that

$$(S_3 \cup S_4) \cap L = \emptyset \quad \text{and} \quad (S_3 \cap S_4) \cap \big(V(\mathcal{C}) - (K \cup L)\big) = \emptyset. \tag{3.6}$$

Without loss of generality, let us assume that rows $A_{1.}, \dots, A_{4.}$ correspond to edges $S_1, \dots, S_4$. Let us call H the submatrix formed by these four rows and let $\bar{y} = \frac{1}{2}(e_1 + e_2 + e_3 + e_4)$. By (3.5) and (3.6) we have

$$\bar{y}A = \frac{1}{2}eH = \frac{1}{2}(A_{1.} + A_{2.} + A_{3.} + A_{4.}) \le e. \tag{3.7}$$

Since $\bar{y}e = 2$, $\bar{y}$ is an optimum solution to $\max\{ye : yA \le e, y \ge 0\}$. By Proposition 2.2 (ii) we get:

$$\frac{1}{2}eH = \bar{y}A = e. \tag{3.8}$$

For every unordered pair (k, l) with $k, l \in \{1, \dots, 4\}$ and $k \ne l$, we associate an index $r(k, l)$ as follows: $r(1, 2) = 1$, $r(3, 4) = 2$, $r(1, 4) = 3$, $r(2, 3) = 4$, $r(1, 3) = 5$, $r(2, 4) = 6$. Also let

$$I_{r(k,l)} = \{i \in V(\mathcal{C}) : i \in S_k \cap S_l\}$$

Note that (3.8) implies that every $i \in V(\mathcal{C})$ belongs to exactly two of $S_1, \dots, S_4$. It follows that $I_1, \dots, I_6$ are all pairwise disjoint and that $I_1 \cup \dots I_6 = V(\mathcal{C})$. Finally, since none of S_1 to S_4 are pairwise disjoint (otherwise $\mathcal{C}$ would pack), we have that $I_{r(k,l)}$ are all nonempty. $\square$

4. New Families

In this section, we construct ideal minimally non packing clutters $\mathcal{C}$ with $\tau(\mathcal{C}) = 2$. By Theorem 1.5, these clutters have the Q_6 property. Thus $V(\mathcal{C})$ can be partitioned into $I_1, \dots, I_6$ and there exist edges $S_1, \dots, S_4$ in $\mathcal{C}$, as defined in Section 3. Without loss of generality we can reorder the vertices in $V(\mathcal{C})$ so that elements in I_k preceed elements in I_p when $k < p$.

Given a set $\mathcal{P}$ of p elements, let $\mathcal{H}_p$ denote the $\big((2^p - 1) \times p\big)$ matrix whose rows are the characteristic vectors of the nonempty subsets of $\mathcal{P}$, and let $\mathcal{H}_p^*$ be its complement, i.e. $\mathcal{H}_p + \mathcal{H}_p^* = J$.

For each $r, t \geq 1$ let $|I_1| = |I_2| = r$, $|I_3| = |I_4| = t$ and $|I_5| = |I_6| = 1$. We call $Q_{r,t}$ the clutter corresponding to the matrix

$$A(Q_{r,t}) = \begin{array}{c} \begin{array}{cccccc} \;I_1 & \;\;I_2 & \;\;I_3 & \;\;I_4 & \;I_5 & I_6 \end{array} \\ \left[\begin{array}{c|c|c|c|c|c} \mathcal{H}_r & \mathcal{H}_r^* & J & \mathbf{0} & 1 & 0 \\ \hline \mathcal{H}_r^* & \mathcal{H}_r & \mathbf{0} & J & 1 & 0 \\ \hline J & \mathbf{0} & \mathcal{H}_t^* & \mathcal{H}_t & 0 & 1 \\ \hline \mathbf{0} & J & \mathcal{H}_t & \mathcal{H}_t^* & 0 & 1 \end{array}\right] \end{array}$$

where J denotes a matrix filled with ones. The rows are partitioned into four sets that we denote respectively by $T(3,5)$, $T(4,5)$, $T(1,6)$, $T(2,6)$. The indices k, l for a given family indicate that the set $I_k \cup I_l$ is contained is every element of the family. Note that the edge S_1 occurs in $T(3,5)$, S_2 in $T(1,6)$, S_3 in $T(4,5)$ and S_4 in $T(2,6)$.

Since $\mathcal{H}_1$ contains only one row, we have $Q_{1,1} = Q_6$ and $Q_{2,1}$ is given by

$$A(Q_{2,1}) = \left[\begin{array}{cc|cc|c|c|c|c} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ \hline 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ \hline 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ T(3,5) \\ \\ \\ T(4,5) \\ \\ T(1,6) \\ T(2,6) \end{array}$$

The proof of the next proposition is straightforward but tedious (see Guenin [8] for details).

Proposition 4.1. *For all $r, t \geq 1$, the clutter $Q_{r,t}$ is ideal and minimally non packing.*

The clutter $\mathcal{D}$ obtained by *duplicating* element $j \in V(\mathcal{C})$ of $\mathcal{C}$ is defined by: $V(\mathcal{D}) = V(\mathcal{C}) \cup \{j'\}$ where $j' \notin V(\mathcal{C})$ and $E(\mathcal{D}) = \{S : j \notin S \in E(\mathcal{C})\} \cup \{S \cup \{j'\} : j \in S \in E(\mathcal{C})\}$. Let $\alpha(k)$ be the mapping defined by: $\alpha(1) = 2$, $\alpha(2) = 1$, $\alpha(3) = 4$, $\alpha(4) = 3$, $\alpha(5) = 6$, $\alpha(6) = 5$.

Suppose that, for $k \in \{1, .., 6\}$, we have that I_k contains a single element $j \in V(\mathcal{C})$. Then j belongs to exactly two of $S_1, \ldots, S_4$. These two edges are of the form $\{j\} \cup I_r \cup I_t$ and $\{j\} \cup I_{\alpha(r)} \cup I_{\alpha(t)}$. We can construct a new clutter $\mathcal{C} \otimes j$ by duplicating element j in $\mathcal{C}$ and including in $E(\mathcal{C} \otimes j)$ the edges:

$$\begin{aligned} &\{j\} \cup I_{\alpha(j)} \cup I_r \cup I_t, \\ &\{j'\} \cup I_{\alpha(j)} \cup I_{\alpha(r)} \cup I_{\alpha(t)}. \end{aligned} \tag{4.9}$$

Since the $\otimes$ construction is commutative we denote by $\mathcal{C} \otimes \{k_1, \ldots, k_s\}$ the clutter $(\mathcal{C} \otimes k_1) \ldots \otimes k_s$. For Q_6, we have $I_1 = \{1\} = S_1 \cap S_2$ and $\{1\} \cup I_{\alpha(1)} \cup I_3 \cup I_5 = \{1, 2, 3, 5\}$ and finally $\{1'\} \cup I_{\alpha(1)} \cup I_{\alpha(3)} \cup I_{\alpha(5)} = \{1', 2, 4, 6\}$. Thus

$$A(Q_6 \otimes 1) = \left[\begin{array}{cc|c|c|c|c|c} 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ \hline 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 1 \end{array}\right]$$

Again, we refer the reader to Guenin's dissertation [8] for a proof of the next result.

Proposition 4.2. *Any clutter obtained from Q_6 and the $\otimes$ construction is ideal and minimally non packing.*

The clutter $Q_6 \otimes \{1,3,5\}$ was found by Schrijver [14] as a counterexample to a conjecture of Edmonds and Giles on dijoins. Prior to this work, Q_6 and $Q_6 \otimes \{1,3,5\}$ were the only known ideal mnp clutters. Eleven clutters can be obtained using Proposition 4.2. There are also examples that do not fit any of the above constructions, as shown by the following ideal mnp clutter.

$$A(\mathcal{C}) = \left[\begin{array}{cc|cc|c|c|c|c} 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ \hline 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 \\ \hline 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 \end{array}\right]$$

5. Non ideal minimally non packing clutters

As mentioned in Remark 1.4, a non ideal mnp clutter is always mni. The following is a result of Bridges and Ryser [2]:

Theorem 5.1. *Let $\bar{A}$, $\bar{B}$ be $n \times n$ 0,1 matrices satisfying $\bar{A}\bar{B}^T = J + dI$, where $d \geq 1$. Then*

(i) Columns and rows of $\bar{A}$ (resp. $\bar{B}$) have exactly r (resp. s) ones with $d = rs - n$.
(ii) $\bar{A}\bar{B}^T = \bar{A}^T\bar{B}$
(iii) $\bar{A}^T(\bar{B}.j) = e + de_j$

Note that, in Theorem 5.1, Property (iii) follows from the equality $\bar{A}^T\bar{B} = J + dI$. The next remark collects known properties of mni matrices [10], [13], [17]. Note that these properties follow readily from Theorem 1.7 (iii) and Theorem 5.1: Point (i) follow from the unicity of the core, and Point (ii) then follows from Point (i). Point (iii) is implied by the fact that the core is a square matrix. Finally, Point (iv) is nothing more than a rewording of Theorem 5.1 (iii).

Remark 5.2. Let A be an $m \times n$ mni matrix, $B = b(A)$, $r = \tau(B)$ and $s = \tau(A)$. Let $\bar{A}$ (resp. $\bar{B}$) be the core of A (resp. B) and let $Q(A)$ denote $\{x \geq 0 : Ax \geq e\}$.

(i) $Q(A)$ (resp. $Q(B)$) has a unique fractional extreme point $\frac{1}{r}e$ (resp. $\frac{1}{s}e$).

(ii) $\min\{ex : Ax \geq e, x \geq 0\} \notin Z$.

(iii) Rows in A (resp. B) that are not rows of $\bar{A}$ (resp. $\bar{B}$) have at least $r+1$ (resp. $s+1$) ones.

(iv) $\bar{A}/j$ (resp. $\bar{B}/j$) packs with s (resp. r) rows whose indices are given by the incidence vector of column j of $\bar{B}$ (resp. $\bar{A}$).

Given a mni clutter $\mathcal{C}$, we will denote by $\bar{\mathcal{C}}$ the core of $\mathcal{C}$. Let $\mathcal{D} = b(\mathcal{C})$ and let L be the set corresponding to the i^{th} row of $A(\bar{\mathcal{C}})$. By Theorem 1.7 (iii), L intersects all sets of $E(\bar{\mathcal{D}})$ exactly once except for the i^{th} row of $A(\bar{\mathcal{D}})$ that is intersected $rs - n + 1 \geq 2$ times. This particular row is called the *mate* of L.

Now we give a proof of Theorem 1.8 stating that if $\mathcal{C} \neq \mathcal{J}_t$ is a mni clutter with $rs > n+1$, then $\mathcal{C}$ is not minimally non packing.

Proof of Theorem 1.8: Let $L \in E(\bar{\mathcal{C}})$ and let U be its mate. We define $I = (L - U) \cup \{i\}$ where i is any element in $L \cap U$.

Claim 1. $\tau(\bar{\mathcal{C}} \setminus I) \geq s - 1$.

Proof of Claim: By contradiction, suppose there is a set $T \in E\big(b(\bar{\mathcal{C}} \setminus I)\big)$ with $|T| \leq s - 2$. Let j be any element in $U - \{i\}$. By Remark 5.2 (iv), L is among the s disjoint sets of $E(\bar{\mathcal{C}}/j)$. Since $I \subseteq L$, there are $s-1$ sets in $E(\bar{\mathcal{C}} \setminus I)$ that intersect only in column j. Therefore, $|T| \leq s-2$ implies $j \in T$. By symmetry among the members of $U - \{i\}$, it follows that $U - \{i\} \subseteq T$. So in particular $|T| \geq s - 1$, a contradiction. $\diamond$

Suppose $\mathcal{C} \setminus I$ packs. Then, since $\tau(\mathcal{C} \setminus I) \geq \tau(\bar{\mathcal{C}} \setminus I)$, it follows from Claim 1 that there must be $s-1$ disjoint sets $\{L_1, \ldots, L_{s-1}\}$ in $E(\mathcal{C} \setminus I)$.

Claim 2. *None of* $\{L_1, \ldots, L_{s-1}\}$ *are in* $E(\bar{\mathcal{C}})$.

Proof of Claim: By contradiction, suppose that L_1 is in $E(\bar{\mathcal{C}})$. Let U_1 be its mate and $q = rs - n + 1 \geq 3$. We have:

$$\tau\big[\mathcal{C} \setminus (I \cup L_1)\big] \quad \leq \quad |U_1 - L_1| = |U_1| - q = s - q \leq s - 3$$

where the first inequality follows from the fact that $b(\mathcal{C} \setminus (I \cup L_1)) = b(\mathcal{C})/(I \cup L_1)$. But $\{L_2, \ldots, L_{s-1}\}$ are disjoint sets of $E\big(\mathcal{C} \setminus (I \cup L_1)\big)$, a contradiction. $\diamond$

By Remark 5.2 (iii), all sets in $E(\mathcal{C}) - E(\bar{\mathcal{C}})$ have cardinality at least $r + 1$. Moreover $\{L_1, \ldots, L_{s-1}\}$ do not intersect I. Therefore we must have:

$$(r+1)(s-1) \leq n - |I| = rs - q + 1 - (r - q + 1) = rs - r$$

Thus $s \leq 1$, a contradiction. $\square$

6. Weakly binary clutters

Let us first show that binary clutters are weakly binary (see Section 1). Given two sets S_1 and S_2, $S_1 \Delta S_2$ denotes the symmetric difference of S_1 and S_2, i.e. $(S_1 \cup S_2) - (S_1 \cap S_2)$. If the clutter $\mathcal{C}$ is binary, then for any k sets $S_1, \ldots, S_k$ with k odd, the set $S_1 \Delta \ldots \Delta S_k$ contains a set of $E(\mathcal{C})$ [16]. Given $\mathcal{C}$ that contains an odd hole C_k^2, let $S_1, \ldots, S_k$ be the k sets in $E(\mathcal{C})$ corresponding to $E(C_k^2)$. If $\mathcal{C}$ is binary, then C_k^2 has a non intersecting set $S \subseteq S_1 \Delta \ldots \Delta S_k$. Since minors of binary clutters are again binary [16], it follows that binary clutters are indeed weakly binary. The inclusion is strict however, since P_4 defined as $V(P_4) = \{1,2,3,4\}$ and $E(P_4) = \big\{\{1,2\},\{2,3\},\{3,4\}\big\}$ is weakly binary but not binary.

In the remainder, we prove Theorem 1.14, stating that if $\mathcal{C}$ is weakly binary and minimally non MFMC, then $\mathcal{C}$ is ideal. To prove this result, we need the following theorem. Given a family of sets $H \subseteq E(\mathcal{C})$ we will denote by $\mathcal{C} - H$ the clutter defined by $V(\mathcal{C} - H) = V(\mathcal{C})$ and $E(\mathcal{C} - H) = E(\mathcal{C}) - E(H)$.

Theorem 6.1. *Let $\mathcal{C} \neq \mathcal{J}_t$ be a mni clutter with $rs = n + 1$. Then $\forall i \in V(\mathcal{C})\ \exists H \subset \{S \in E(\mathcal{C}) : i \in S\}$ such that there is a minor $\mathcal{D}$ of $\mathcal{C} - H$ with*

1. *$i \in V(\mathcal{D})$ and*
2. *$\mathcal{D}$ contains an odd hole C_k^2 with $V(\mathcal{D}) = V(C_k^2)$.*

To illustrate this theorem, consider $b(C_5^2)$. We have $E\big(b(C_5^2)\big) = \big\{\{1,3,5\},\{1,2,4\},\{2,3,5\},\{1,3,4\},\{2,4,5\}\big\}$. For $i = 1$, let $H = \big\{\{1,3,4\}\big\}$. Then $E\big([b(C_5^2) - H]/\{3,4\}\big) = \big\{\{1,2\},$

$\{2,5\},\{1,5\}\}$. We will need the following definition for the proof. A clutter $\mathcal{C}$ is *bicolorable* if there is a partition of $V(\mathcal{C})$ into V_1 and V_2 such that every element of $E(\mathcal{C})$ intersects V_1 and V_2.

Proof of Theorem 6.1: Let $\mathcal{B} = b(\mathcal{C})$, and let $\bar{\mathcal{C}}$ (resp. $\bar{\mathcal{B}}$) denote the core of $\mathcal{C}$ (resp. $\mathcal{B}$). Let $i \in V(\mathcal{C})$. Moreover, let $L_1,\dots,L_r$ be the edges in $E(\bar{\mathcal{C}})$ that contain i. Finally, for $j = 1,\dots,r$, let U_j be the mate of L_j. Then, by Remark 5.2 (iv), $U_j \cap U_\ell \subseteq \{i\}$ if $j \neq \ell$ and, by Theorem 5.1 (iii), exactly two U_j's, say U_1 and U_2, contain $\{i\}$, since $rs = n+1$. Let $I_c = \bigcup_{j=3}^r U_j$ and $H = \{S \in E(\mathcal{C}) : i \in S\} - \{L_1, L_2\}$. We define $\mathcal{D}' = (\bar{\mathcal{C}} - H)/I_c$.

Claim 1. *Sets in $E(\mathcal{D}')$ have cardinality 2.*

Proof of Claim: Let L be any set in $E(\bar{\mathcal{C}} - H)$. We want to show that $|L - \cup_{j=3}^r U_j| = 2$. Since the complement of $\cup_{j=3}^r U_j$ is $U_1 \cup U_2$, this is equivalent to show that $|L \cap (U_1 \cup U_2)| = 2$. Suppose $i \notin L$. Then L is not a mate of U_1 or U_2. Thus $|L \cap U_1| = |L \cap U_2| = 1$. Since $U_1 \cap U_2 \subseteq \{i\}$ we have $|L \cap (U_1 \cup U_2)| = 2$. Now suppose $i \in L$. By definition of H, $L = L_1$ or $L = L_2$. Without loss of generality we can assume $L = L_1$. Now $|L_1 \cap (U_1 \cup U_2)| = |(L_1 \cap U_1) \cup (L_1 \cap U_2)| = |(L_1 \cap U_1) \cup \{i\}| = |L_1 \cap U_1| = 2$, where the last equality follows from the fact that L_1 is the mate of U_1. $\diamond$

Claim 2. *There is no set T such that $|T \cap L| = 1, \forall L \in E(\bar{\mathcal{C}})$.*

Proof of Claim: By Theorem 5.1 (iii), for any $j \in V(\mathcal{C})$ there are sets $S_1^j,\dots,S_s^j \in E(\bar{\mathcal{C}})$ that intersect only in j. Moreover, $\bigcup_{i=1}^s S_i^j = V(\bar{\mathcal{C}})$ and exactly $rs - n = 2$ of those sets, say S_1^j, S_2^j contain j. By choosing $j \in T$ we obtain that $T - \{j\}$ does not intersect $S_1^j \cup S_2^j$ and that $T - \{j\}$ intersects each $S_3^j \dots S_s^j$ at most once. Hence $|T| \leq s-1$. By choosing $j \notin T$, we have $|T| \geq s$ since T intersects the sets $S_1^j,\dots,S_s^j$, a contradiction. $\diamond$

Claim 3. *$\mathcal{D}'$ is not bicolorable.*

Proof of Claim: Suppose that $\mathcal{D}'$ is bicolorable. Let T, T' be the corresponding partition of $V(\mathcal{D}')$. Without loss of generality we can assume that $i \in T$. Let L be any set of $E(\bar{\mathcal{C}})$. We will show that $|T \cap L| = 1$ thereby contradicting Claim 2. Suppose $L - I_c \in E(\mathcal{D}')$. By Claim 1, $|L - I_c| = 2$. Since $T \cap (L - I_c)$ and $T' \cap (L - I_c)$ are both non empty, we must have $1 = |T \cap (L - I_c)| = |T \cap L|$.

Thus we can assume that $L - I_c \notin E(\mathcal{D}')$, i.e. that $i \in L$ and $L \neq L_1, L \neq L_2$. Therefore L is the mate of some set U_j with $j \geq 3$. But then, as $T = T \cap (U_1 \cup U_2)$ and $L \cap (U_1 \cup U_2) = \{i\}$, we have $L \cap T = L \cap (U_1 \cup U_2) \cap T = \{i\}$. $\diamond$

Claim 4. *$\mathcal{D}'$ contains an odd hole C_k^2.*

Proof of Claim: By Claim 1, all elements of $E(\mathcal{D}')$ have cardinality 2. Therefore $M(\mathcal{D}')$ can be viewed as the edge-vertex incidence matrix of a graph G. Since $\mathcal{D}'$ is not bicolorable G cannot be bipartite. Therefore G has a vertex induced subgraph G' that is a triangle or an odd hole. In both cases G' corresponds to an odd hole C_k^2 contained in $\mathcal{D}'$. $\diamond$

Claim 5. *Every edge in $(\mathcal{C} - H)/I_c$ has cardinality at least 2.*

Proof of Claim: By Claim 1 it is sufficient to show that sets $L \in E(\mathcal{C} - H) - E(\bar{\mathcal{C}} - H)$ satisfy $|L \cap (U_1 \cup U_2)| \geq 2$. Since $L \notin E(H) \cup E(\bar{\mathcal{C}})$ we have $i \notin L$. The result then follows from the fact that $(U_1 - \{i\}) \cap (U_2 - \{i\}) = \emptyset$. $\diamond$

Let $I_d = V(\mathcal{D}') - V(C_k^2)$ and let $\mathcal{D} = (\mathcal{C} - H)/I_c \setminus I_d$. By Claim 4, $\mathcal{D}' = (\bar{\mathcal{C}} - H)/I_c$ contains an odd hole C_k^2. By Claim 5, the sets corresponding to the odd hole are in the clutter $(\mathcal{C} - H)/I_c$. Hence $\mathcal{D}$ satisfies item (2) in the statement of the theorem. The next claim will show item (1).

Claim 6. $i \in V(\mathcal{D})$

Proof of Claim: Suppose $i \notin V(\mathcal{D})$. Then $i \in I_d$ and thus, by the choice of H, we have that $\mathcal{D} = (\mathcal{C} - H)/I_c \setminus I_d = \mathcal{C}/I_c \setminus I_d$, i.e. $\mathcal{D}$ is a minor of $\mathcal{C}$. But $\frac{1}{2}e$ is a fractional extreme point of $\{x \geq 0 : A(\mathcal{D}) \geq e\}$, a contradiction with $\mathcal{C}$ mni. $\square$

We are now ready to prove the main result of this section.

Proof of Theorem 1.14: Suppose $\mathcal{C}$ is not ideal. From Remark 1.4, we have that $\mathcal{C}$ is mni. $\mathcal{C} \neq \mathcal{J}_t$ since $\mathcal{J}_t$ is not weakly binary. Indeed the odd hole of $\mathcal{J}_t$ defined by the sets $\{1, \ldots, t\}, \{0, 1\}, \{0, 2\}$ does not have a non intersecting set. By Theorem 1.8, we must also have $rs = n + 1$.

Consider $\mathcal{D} = (\mathcal{C} - H) \setminus I_d/I_c$ in Theorem 6.1. Note that $\mathcal{C} \setminus I_d$ contains the odd hole C_k^2. Since $\mathcal{C}$ is weakly binary, there is a non intersecting set S of C_k^2 in $E(\mathcal{C} \setminus I_d)$. Here $S \cap \left[V(C_k^2) \cup I_d\right] = \emptyset$. Since $i \notin S$, we have $S \notin E(H)$ and therefore $S - I_c$ contains an edge of $\mathcal{D}$. But since $V(\mathcal{C}) = V(C_k^2) \cup I_c \cup I_d$ we must have $S - I_c = \emptyset$, a contradiction. $\square$

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